Projective Geometry and the Origins of the Dirac Equation

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"the solution came rather, I would say, out of the blue" (Dirac, 1977)

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- The Dirac Equation
  - A role for projective geometry?
  - Archival evidence

Dirac's Education

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Dirac had a background in pure mathematics.

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Dirac was keen to speak about his fondness for projective geometry. Is there a connection to his work in quantum mechanics?

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- 1927 'The Physical Interpretation of Quantum Mechanics'

Definition of q-numbers

At present one can form no picture of what a q-number is like. One cannot say that one q-number is greater or less than another. All one knows about q-numbers is that if z_1 and z_2 are two q-numbers, or one q-number and one c-number, there exist the numbers $z_1 + z_2$, $z_1 z_2$, $z_2 z_1$, which will in general be q-numbers but may be c-numbers. One knows nothing of the processes by which the numbers are formed except that they satisfy all the ordinary laws of algebra, excluding the commutative law of multiplication, *i.e.*,

$$z_1 + z_2 = z_2 + z_1,$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3),$$

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3, \qquad (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3,$$

and if

$$z_1 z_2 = 0,$$

either

$$z_1 = 0 \quad \text{or} \quad z_2 = 0\,;$$

but

$$z_1 z_2 \neq z_2 z_1,$$

in general, except when z_1 or z_2 is a c-number.

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4. Dirac's quantum algebra was essentially geometrical (Rechenberg 1987)

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Dirac's primary role for projective geometry was as a means of visualization for Minkowski space and Lorentz transformations NOT *q*-numbers.

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So what is the connection to projective geometry?

Projective Geometry and Special Relativity

"Four dimensions were very popular then for the geometrists to work with. It was all done with the notions of projective geometry rather than metrical geometry. So I became very familiar with that kind of mathematics in that way. I've found it useful since then in understanding the relations which you can have in Minkowskis space. You can picture all the directions in Minkowski space as the points in a three-dimensional vector space. I always used these geometri- cal ideas for getting clear notions about relationships in relativity although I didn't refer to them in my published works." (ibid.)

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> *"No. It doesn't connect at all with non-commutative*

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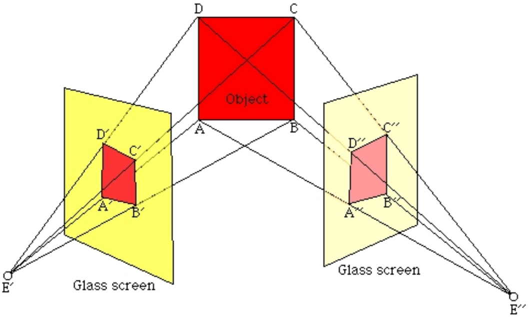
Did Dirac use projective geometry in his search for the Dirac equation?

What is Projective Geometry?

The study of geometrical properties invariant under projection.



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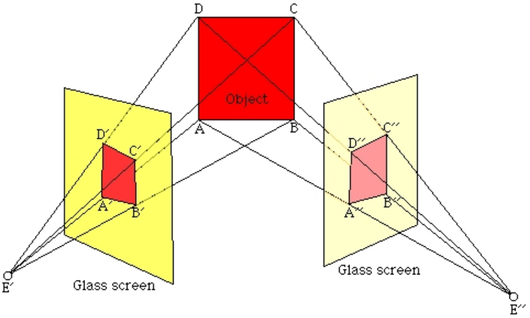

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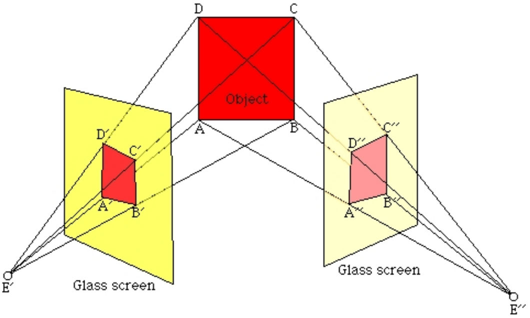

- Any two lines meet at a unique point.
- Parallel lines meet at a point *at infinity*.

An Illustration

Homogeneous Co-ordinates

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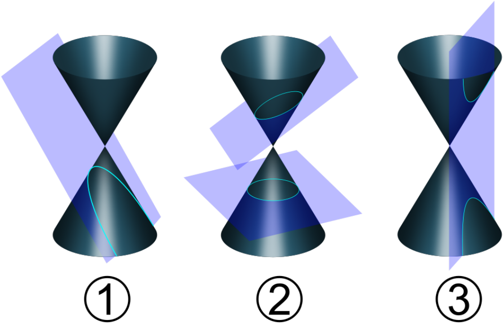
- The points $(x_1, x_2, 0)$ form the *line at infinity*; approached from either direction.

Conic Sections

Conic sections 1) Parabolae, 2) Circles and Ellipses, 3) Hyberbolae.



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- Related by projections that swap points, so not distinguished projectively.
- Quadrics generalise the conic to higher dimensional spaces.

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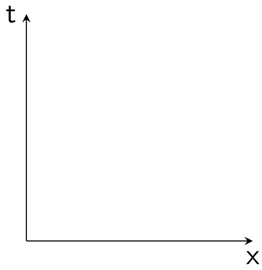

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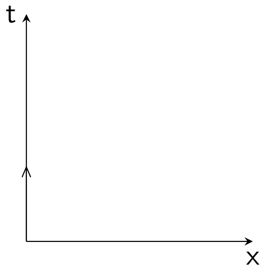

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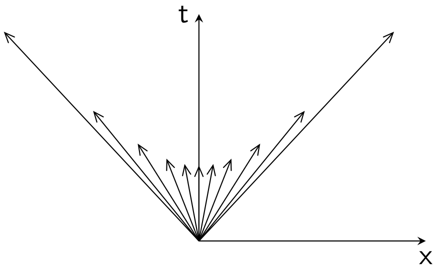

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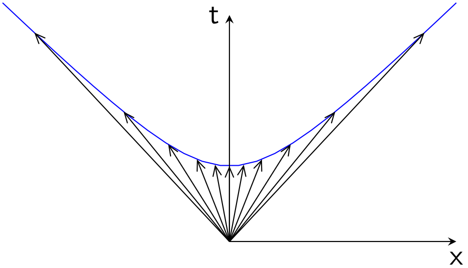

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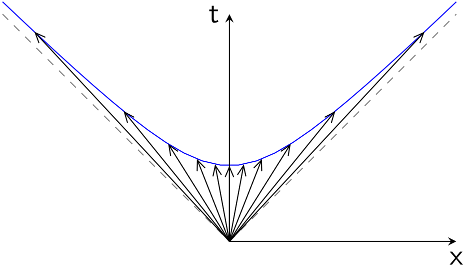

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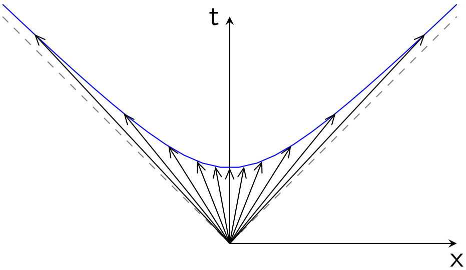

Lorentz transformation leaves hyperboloid invariant.

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- Lorentz transformations leave the *absolute quadric* invariant.

The Absolute Quadric

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In the case of Minkowski space, defined by the Minkowski metric:

$$\eta_{\mu\nu}x^{\mu}x^{\nu} = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$$

Dirac Speaks

"if we just think in terms of this hyperplane at infinity, we have a three-dimensional space. Talking of a four dimensional space is something that is hard to imagine, but we can't really imagine it. We talk about it as though we could, but when we are concerned just with directions, the things in the space of physics, we can represent them all in terms of a three-dimensional space according to the methods of projective geometry. We have a three-dimensional projective space in which there is an absolute quadric." (Dirac, 1972)

Dirac's Task

Dissatisfied with Klein-Gordon equation

$$\left[\left(ih\frac{\partial}{c\partial t}+\frac{e}{c}A_0\right)^2+\sum_r\left(-ih\frac{\partial}{\partial x_r}+\frac{e}{c}A_r\right)^2+m^2c^2\right]\psi=0.$$

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Dirac's Story

"Playing around with mathematics" he noticed the "very pretty mathematical result" (Dirac, 1972)

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> *"It took me quite a while, studying over this dilemma ..."*

Came to realize that 4×4 matrices would suffice.

The Solution

The symmetry between p_0 and p_1, p_2, p_3 required by relativity shows that, since the Hamiltonian we want is linear in p_0, it must also be linear in p_1, p_2 and p_3. Our wave equation is therefore of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\,\psi = 0, \tag{4}$$

Equation (4) leads to

$$\begin{aligned} 0 &= (-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\,\psi \\ &= [-p_0{}^2 + \Sigma\,\alpha_1{}^2 p_1{}^2 + \Sigma\,(\alpha_1\alpha_2 + \alpha_2\alpha_1)\,p_1 p_2 + \beta^2 + \Sigma\,(\alpha_1\beta + \beta\alpha_1)\,p_1]\,\psi, \end{aligned} \tag{5}$$

where the Σ refers to cyclic permutation of the suffixes 1, 2, 3. This agrees with (3) if

$$\left.\begin{array}{ll} \alpha_r{}^2 = 1, & \alpha_r\alpha_s + \alpha_s\alpha_r = 0 \quad (r \neq s) \\ \beta^2 = m^2c^2, & \alpha_r\beta + \beta\alpha_r = 0 \end{array}\right\} \quad r, s = 1, 2, 3.$$

If we put $\beta = \alpha_4 mc$, these conditions become

$$\alpha_\mu{}^2 = 1 \qquad \alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 0 \ (\mu \neq \nu) \qquad \mu, \nu = 1, 2, 3, 4. \tag{6}$$

We can suppose the α_μ's to be expressed as matrices in some matrix scheme,

The Solution

We must now find four matrices α_μ to satisfy the conditions (6). We make use of the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which Pauli introduced* to describe the three components of spin angular momentum. These matrices have just the properties

$$\sigma_r^2 = 1 \qquad \sigma_r\sigma_s + \sigma_s\sigma_r = 0, \qquad (r \neq s), \tag{7}$$

that we require for our α's. We cannot, however, just take the σ's to be three of our α's, because then it would not be possible to find the fourth. We must extend the σ's in a diagonal manner to bring in two more rows and columns, so that we can introduce three more matrices ρ_1, ρ_2, ρ_3 of the same form as σ_1, σ_2, σ_3, but referring to different rows and columns, thus :—

$$\sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \rho_2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \quad \rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If we now take

$$\alpha_1 = \rho_1\sigma_1, \qquad \alpha_2 = \rho_1\sigma_2, \qquad \alpha_3 = \rho_1\sigma_3, \qquad \alpha_4 = \rho_3,$$

all the conditions (6) are satisfied, *e.g.*,

$$\alpha_1^2 = \rho_1\sigma_1\rho_1\sigma_1 = \rho_1^2\sigma_1^2 = 1$$
$$\alpha_1\alpha_2 = \rho_1\sigma_1\rho_1\sigma_2 = \rho_1^2\sigma_1\sigma_2 = -\rho_1^2\sigma_2\sigma_1 = -\alpha_2\alpha_1.$$

Dirac's Method

John Slater:

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Can we say more about Dirac's process of discovery than an idea "out of the blue?"

In particular, did he use projective geometry?

Comments

$$F\psi \equiv \left[\left(\alpha_{5\mu} + i\alpha_{6\mu}\right)\left(d_\mu + iA_\mu\right) + mc\right]\psi = 0 \qquad (1)$$

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- Dirac has found here the general form of the equation he seeks.
- No explicit (anti)-commutation relations.
- Nature of the α's unclear, but not c-numbers.

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Where did they come from?

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- In Klein (1870) co-ordinates, this has a general quadratic form.

Did Projective Geometry Lead to the Dirac Equation?

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Was he guided by geometrical, not algebraic reasoning?

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- Tries to find 3×3 matrices with these properties.
- Finds suitable 4×4 matrices - leads to α's on p. 7.

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- Projective geometry was primarily a means for visualisation of Minkowski space.
- Definite mathematical correspondence and clear role.
- Dirac used projective geometry in his search for the Dirac equation.

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- Dirac was right: he did not consider the Pauli equation and spin, although he did try two component wave functions.
- The realization that α's were analogous to Pauli matrices led straight to the solution - no delay.
- Dirac did consider 3×3 matrices (*contra* Mehra and Rechenberg, 2000).